OPTIMAL CONTROL AND STATE CONSTRAINTS

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INTRODUCTION

Consider the following dynamic system:

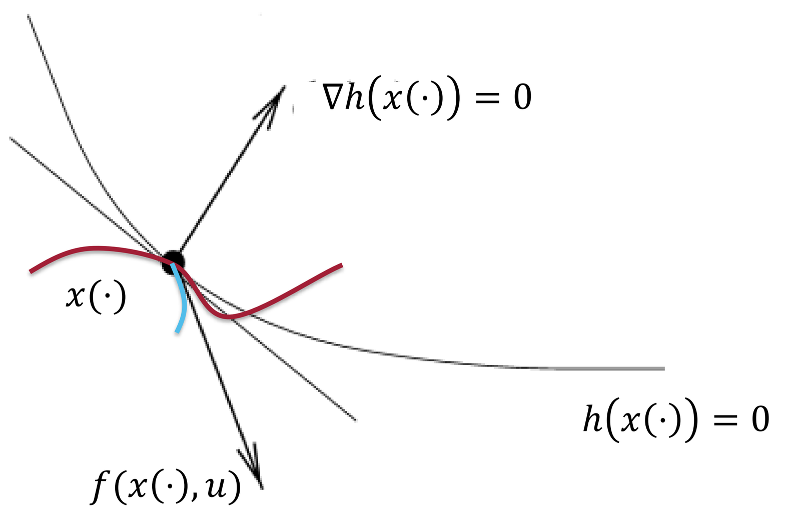
$$\begin{cases} \dot{x} = f(x,u) = f_0(x) + u_1 f_1(x) + u_2 f_2(x) \\ u \in \mathcal{U} \\ x(0) = x_0 \end{cases} \qquad (1)$$

where $\mathcal{U} = \{(u_1,u_2) \in \mathbb{R}^2 : {u_1}^2 + {u_2}^2 \leq 1\}$ and f_0, f_1, f_2, h are regular enough, under the state constraint $x(t) \in C := \{x \in \mathbb{R}^n : h(x) \leq 0\}$. In Optimal Control Theory with state constraints we are not always able to find trajectories which are both optimal (in the sense of optimising a functional) and do not violate the constraints. Let us consider a reference trajectory $\bar{x}(\cdot)$ on a time interval $[0,T]$ which corresponds to an optimal solution, but at the same time may violate the constraint. Recent papers, which cover this problem, tell us that, for dynamic like (1) but without the drift component (f_0), there exists a trajectory $y(\cdot)$, such that it does not violate constraints, and at the same time, is not so far from the reference one. It is called Neighbouring Feasible Trajectory. Our goal consists in obtaining similar results for the systems with a not controllable influence (i.e., with the drift component).

MAIN RESULTS AND APROACH

Let us observe one of the most important concepts in constructing Neighbouring Feasible Trajectories is the concept, of Inward Pointing Condition (IPC) at $x(\cdot)$. Namely, there exists $\gamma > 0$, such that:

$$\nabla h(x(\cdot)) \cdot f(x(\cdot),u) \leq -\gamma.$$



This condition tells us that for a reference trajectory (which is the red one), which possibly violates constraints, it is possible to find another feasible trajectory (which is the blue one), which is close enough to the reference one, but at the same time, does not violate constraints. That is, there exists a control that can pull the trajectory away from the state constrained boundary in a neighbourhood of the initial time.

But we have to remember, that a part of the reference trajectory (say the initial point x_0) may lie in the singular set $\mathbb{S}$, where the (IPC) does not hold. Hence, we need to have a new approach which may help to deal with such cases. The approach consists in formulating another condition which replaces (IPC). Such condition simply states, that for any $x \in \mathbb{S}$ the Lie bracket $[f_1, f_2](x)$ of the functions f_l and f_m is not tangent to the constraints set, i.e.

$$\nabla h(x) \cdot [f_1, f_2](x) \neq 0.$$

We have shown that with this condition and under some assumptions on f_0, f_1, f_2, h and $\boldsymbol{S}(x)$ it is possible to find a neighbouring feasible trajectory to the reference one. As an application of the result, we have also obtained continuity of a value function on C.

DEFINITIONS AND ASSUMPTIONS

- We say that a trajectory $y(\cdot)$ is Neighbouring Feasible to the reference one $\bar{x}$ on the time interval $[0,T]$, if for any $\epsilon > 0$ and for $d := max_{t\in[0,T]}\{\max\{h(\bar{x}(t)), 0\}\}$ the following inequalities hold:

$$||y(\cdot) - \bar{x}(\cdot)||_{L^\infty([0,T])} \leq \epsilon + K\sqrt{d},$$

$$h\big(y(t)\big) < 0, \forall t \in (0,T]\ (i.e.\ y(t) \in C^\circ).$$

- We say that the functions f_0, f_1, f_2, h are regular, if f_0, f_1, f_2, h are of class C^2.

- One of the crucial assumption is the assumption on a symmetric quadratic form $\boldsymbol{S}(x)$, which can be interpreted as the curvature matrix at x of functions f_0, f_1, f_2 with respect to the foliation $\{h = c\}$, that the form is negative semi-definite or indefinite.

- We say that a trajectory $y(\cdot)$ is feasible, if it satisfies (1) and does not violate constraints.

- We define the singular set $\mathbb{S}$ as:

$$\mathbb{S} := \{x \in \mathbb{R}^n : \nabla h(x) f_1(x) = \nabla h(x) f_1(x) = 0, \nabla h(x) f_0(x) \leq 0\}$$

BROCKETT DYNAMIC EXAMPLE

Suppose that $q = (x,y,z) \in \mathbb{R}^3$, and consider the constraint function:

$$h(q) = k(x^2 + y^2)^2 + z,$$

where $k \in \mathbb{R}$ a constant. The dynamic of the Brockett non-holonomic integrator has the form (1). The f_0, f_1, f_2 functions have the following representation:

$$f_0(q) = \begin{bmatrix} \eta_1 \\ \eta_2 \\ -\eta_3(x^2+y^2)^2 \end{bmatrix}, f_1(q) = \begin{bmatrix} 1 \\ 0 \\ -y \end{bmatrix}, f_2(q) = \begin{bmatrix} 0 \\ 1 \\ x \end{bmatrix},$$

where η_1, η_2, η_3 are constants as well. We consider a rotational control: $r_\phi(\omega t) = \left({r_\phi}^1(\omega t), {r_\phi}^2(\omega t)\right) = (\cos(\omega t + \phi), \sin(\omega t + \phi))$. Then, we have to check that all the assumptions are verified. From the general analysis of the Brockett dynamic, we have that the regularity assumption holds. Second order (IPC) is also verified. In fact,

$$\nabla h(q) \cdot [f_0, f_1](q) = 4\eta_3 x(x^2+y^2) - \eta_2, \nabla h(q) \cdot [f_0, f_1](q) = 4\eta_3 x(x^2+y^2) + \eta_1, \nabla h(q) \cdot [f_1, f_2](q) = 2.$$

From this, for every η_1, η_2, η_3 we obtain that second order (IPC) holds.

The only thing to do is to check that the symmetric quadratic form is negative semidefinite (or indefinite). Calculating, we have:

$$\boldsymbol{S}(q) = \begin{pmatrix} 0 & 0 & -\frac{\eta_2}{2} \\ 0 & 0 & \frac{\eta_1}{2} \\ -\frac{\eta_2}{2} & \frac{\eta_1}{2} & 0 \end{pmatrix}.$$

From the quadratic form we can see, that by setting $\eta_1 = \eta_2 = 0$ the form is just a zero matrix, while in the general case it is indefinite. Hence, if the dynamic has Brockett's non-holonomic integrator with drift form, we can apply the Neighbouring Feasible Trajectory Theorem, and get the feasible trajectory which is close to the optimal one.

FUTURE WORK

Further, using the approach and the result, we wish to prove some non-degeneracy conditions for Pontryagin's Maximum Principle under state constraints.

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Mathematical Sciences

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